1897–2022
125
TH BINGEN
University of Applied Sciences

Vedic Mathematics

Mental Arithmetic

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Basic Ideas

LITERATURE:

Bharati Krishna Tirtha: *Vedic Mathematics*, 1992
Present from Willi Hahn when I retired from TH Bingen

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1. Vedic Mathematics: Old Indian Mathematics, a lot of formulas to ease mental calculations.
2. They already had the decimal system include the digit 0.
3. Arabic digit are in fact Indian digits!
4. Fibonacci brought them from Arabia to Italy.

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European Method

European Method $499 \cdot 287 = 143213$:

$$
\begin{array}{r}
499 \cdot 287 \\
\hline
998 \\
3992 \\
3493 \\
\hline
143213 \\
\hline\hline
\end{array}
$$

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$$
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499 \cdot 287 \\
\hline
998 \\
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3493 \\
\hline
143213 \\
\hline\hline
\end{array}
$$

Vedic Method $499 \cdot 287 = 143213$:

			4	9	9
			2	8	7
	$4 \cdot 2$	$4 \cdot 8 + 2 \cdot 9$	$4 \cdot 7 + 9 \cdot 8 + 9 \cdot 2$	$9 \cdot 7 + 9 \cdot 8$	$9 \cdot 7$
	6	13	14	6	
1	4	3	2	1	3

Negative Digits

Vedic Method $499 \cdot 287 = 50\bar{1} \cdot 3\bar{1}\bar{3} = 143213$:

			5	0	$\bar{1}$
			3	$\bar{1}$	$\bar{3}$
	$5 \cdot 3$	$5 \cdot \bar{1} + 0 \cdot 3$	$5 \cdot \bar{3} + 0 \cdot \bar{1} + \bar{1} \cdot 3$	$0 \cdot \bar{3} + \bar{1} \cdot \bar{1}$	$\bar{1} \cdot \bar{3}$
		$\bar{1}$			
1	5	$\bar{6}$	$\bar{8}$	1	3
1	4	3	2	1	3

Negative Digits

European Method $499 \cdot 287 = 50\bar{1} \cdot 3\bar{1}\bar{3} = 143213$:

$$
\begin{array}{r}
50\bar{1} \cdot 3\bar{1}\bar{3} \\
\hline
150\bar{3} \\
\bar{5}01 \\
\bar{1}\bar{5}03 \\
\hline
15\bar{6}\bar{8}13 \\
\hline
143213 \\
\hline\hline
\end{array}
$$

Specific Rules for Multiplikation

Basic Formula

$$\begin{aligned} A \cdot B &= (x+a)(x+b) = x(x+a+b) + ab = x(A+b) + ab \\ &= x(B+a) + ab\,. \end{aligned}$$

Specific Rules for Multiplikation — Base 10

Basic Formula

$$A \cdot B = (x+a)(x+b) = x(x+a+b)+ab = x(A+b)+ab = x(B+a)+ab\ .$$

Example: $9 \cdot 8 = 72$

9	-1
8	-2
7	2

Specific Rules for Multiplikation

Base 10

Basic Formula

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Example: $9 \cdot 8 = 72$

9	-1
8	-2
7	2

Example: $19 \cdot 18$

19	9
18	8
27	72

Specific Rules for Multiplikation

Base 10

Basic Formula

$$A \cdot B = (x+a)(x+b) = x(x+a+b)+ab = x(A+b)+ab = x(B+a)+ab\ .$$

Example: $9 \cdot 8 = 72$

9	-1
8	-2
7	2

Example: $19 \cdot 18 = 342$

19	9
18	8
27	72

Numbers Closed to Hundred

Numbers Closed to 100

Take $x = 100$ and calculate the 100^1-position cross-wise, the 100^0-position by multiplication:

Example: $91 \cdot 88$

91	-9
88	-12
79	108

Numbers Closed to Hundred

Numbers Closed to 100

Take $x = 100$ and calculate the 100^1-position cross-wise, the 100^0-position by multiplication:

Example: $91 \cdot 88 = 8008$

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Example: $91 \cdot 88 = 8008$

91	-9
88	-12
79	108

Example: $91 \cdot 112$

91	-9
112	12
103	-108

Numbers Closed to Hundred

Numbers Closed to 100

Take $x = 100$ and calculate the 100^1-position cross-wise, the 100^0-position by multiplication:

Example: $91 \cdot 88 = 8008$

91	-9
88	-12
79	108

Example: $91 \cdot 112 = 10192$

91	-9
112	12
103	-108

$10300 - 108 = 10192.$

Specific Rules for Multiplikation Numbers Closed to Fifty

Formula

$(50+a)(50+b) = 50(50+a+b)+ab = \frac{100(50+a+b)}{2}+ab$:
Base 100, divisor 2

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Formula

$(50+a)(50+b) = 50(50+a+b) + ab = \frac{100(50+a+b)}{2} + ab$:

Base 100, divisor 2

Example: $48 \cdot 53$

48	-2
53	3
51	-6

Specific Rules for Multiplikation

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Formula

$(50+a)(50+b) = 50(50+a+b) + ab = \frac{100(50+a+b)}{2} + ab$:

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Example: $48 \cdot 53 = 2544$

48	-2
53	3
51	-6
25	50-6

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$(50+a)(50+b) = 50(50+a+b) + ab = \frac{100(50+a+b)}{2} + ab$:
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Alternatively:

$(50+a)(50+b) = 50(50+a+b) + ab = 5 \cdot 10(50+a+b) + ab$:
Base 10, factor 5

Example: $48 \cdot 53 = 2544$

48	-2
53	3
51	-6
25	50-6

Example:
$48 \cdot 53$

48	-2
53	3
51	-6
255	-6

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Example: $48 \cdot 53 = 2544$

48	-2
53	3
51	-6
25	50-6

Example:

$48 \cdot 53 = 2550 - 6 = 2544$

48	-2
53	3
51	-6
255	-6

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Example: $32 \cdot 63$

32	-18
63	13
45	-234
225	-234

Auxiliary Calculation:

18	8
13	3
21	24
23	4

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Example: $32 \cdot 63 = 2250 - 234 = 2016$

32	-18
63	13
45	-234
225	-234

Auxiliary Calculation:

18	8
13	3
21	24
23	4

Specific Rules for Multiplikation   Numbers Closed to 250

Example: $260 \cdot 244$

Base 1000, divisor 4:

$$\begin{array}{rr} 260 & 10 \\ 244 & -6 \\ \hline 254 & -60 \\ \hline 63 + \frac{1}{2} & -60 \end{array}$$

Specific Rules for Multiplikation Numbers Closed to 250

Example: $260 \cdot 244 = 63440$

Base 1000, divisor 4:

260	10
244	-6
254	-60
$63 + \frac{1}{2}$	-60

Specific Rules for Multiplikation Numbers Closed to 500

Formula

$(500+a)(500+b) = 500(500+a+b)+ab = \frac{1000(50+a+b)}{2}+ab$:
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Specific Rules for Multiplikation

Numbers Closed to 500

Formula

$(500+a)(500+b) = 500(500+a+b) + ab = \frac{1000(50+a+b)}{2} + ab$:

Base 1000, divisor 2

Example: $505 \cdot 372$

505	5
372	-128
377	-640
188.5	-640
187	500+360

Specific Rules for Multiplikation

Numbers Closed to 500

Formula

$$(500+a)(500+b) = 500(500+a+b) + ab = \frac{1000(50+a+b)}{2} + ab:$$

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Example: $505 \cdot 372 = 187860$

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372	-128
377	-640
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187	500+360

Example: $505 \cdot 372 = 188500 - 640 = 187860$

505	5
372	-128
377	-640
1885	-640

Specific Rules for Multiplikation Products with Identical 10^1-Digits

Formula

$$
\begin{aligned}
A \cdot B &= (10x + a)(10x + b) = 100x^2 + 10x(a + b) + ab \\
&= 10x(10x + a + b) + ab \\
&= 10x(A + b) + ab = 10x(a + B) + ab
\end{aligned}
$$

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Example

- $23 \cdot 28 = 20 \cdot (23 + 8) + 3 \cdot 8 = 20 \cdot 31 + 24 = 644$,
- $23 \cdot 38 = 20 \cdot (23 + 18) + 3 \cdot 18 = 20 \cdot 41 + 54 = 874$,
- $23 \cdot 38 = 30 \cdot (23 + 8) - 7 \cdot 8 = 30 \cdot 31 - 56 = 874$,
- $37 \cdot 44 = 40 \cdot (37 + 4) - 3 \cdot 4 = 40 \cdot 41 - 12 = 1628$.

Specific Rules for Multiplikation — Products with Identical 10^1-Digits

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Specific Rules for Multiplikation Sum of Last Digits Equals Ten

Formula

$$(10x + a)(10x + b) = 100x^2 + 10x(a + b) + ab = 100x(x + 1) + ab$$

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Examples

- $93 \cdot 97 = 100 \cdot 9 \cdot 10 + 21 = 9021$
- $44 \cdot 46 = 100 \cdot 4 \cdot 5 + 24 = 2024$

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- $44 \cdot 46 = 100 \cdot 4 \cdot 5 + 24 = 2024$

Specific Rules for Multiplikation Sum of the las two Digits gets 100

Formula

$$(100x+a)(100x+b) = 10000x^2+100x(a+b)+ab = 10000x(x+1)+ab$$

Specific Rules for Multiplikation Sum of the las two Digits gets 100

Formula

$$(100x+a)(100x+b) = 10000x^2+100x(a+b)+ab = 10000x(x+1)+ab$$

Example

- $123 \cdot 177 = 10000 \cdot 1 \cdot 2 + 23 \cdot 77 = 20000 + 1771 = 21771$

Specific Rules for Multiplikation Third Binomial Formula

Formula

$$(a+b)(a-b) = a^2 - b^2$$

Specific Rules for Multiplikation — Third Binomial Formula

Formula

$$(a+b)(a-b) = a^2 - b^2$$

Examples

- $24 \cdot 26 = (25-1)(25+1) = 625 - 1 = 624$,
- $104 \cdot 96 = (100+4)(100-4) = 10000 - 16 = 9984$.

Specific Rules for Multiplikation

Third Binomial Formula

Formula

$(a+b)(a-b) = a^2 - b^2$

Examples

- $24 \cdot 26 = (25-1)(25+1) = 625 - 1 = 624$,
- $104 \cdot 96 = (100+4)(100-4) = 10000 - 16 = 9984$.

Square Numbers

Numbers Closed to Powers of 10

Formula

$$A^2 = (10^n + a)^2 = 10^n(10^n + a + a) + a^2 = 10^n(A + a) + a^2$$

Examples

13	3
13	3
16	9

  or $13^2 = 10 \cdot (13 + 3) + 9 = 169$
- $107^2 = 100(107 + 7) + 7^2 = 11449$
- $93^2 = 100(93 - 7) + 49 = 8649$

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Square Numbers Numbers closed to a Multiple of a Power of 10

Example: 685^2

Using base 100, factor 7

685	-15
685	-15
670	225

Square Numbers

Numbers closed to a Multiple of a Power of 10

Example: 685^2

Using base 100, factor 7

$$\begin{array}{rr} 685 & -15 \\ 685 & -15 \\ \hline 670 & 225 \end{array}$$

The result is: $7 \cdot 67000 + 225 = 469000 + 225 = 469225$.

Square Numbers

Numbers with Last Digit Equal to 5

Formula

$$(10x + 5)^2 = 100x^2 + 100x + 25 = 100x(x + 1) + 25$$

Square Numbers

Numbers with Last Digit Equal to 5

Formula

$$(10x + 5)^2 = 100x^2 + 100x + 25 = 100x(x + 1) + 25$$

Examples

- $15^2 = 100 \cdot 1 \cdot 2 + 25 = 225$
- $45^2 = 100 \cdot 4 \cdot 5 + 25 = 2025$
- $115^2 = 100 \cdot 11 \cdot 12 + 25 = 13225$
- $135^2 = 100 \cdot 13 \cdot 14 + 25 = 18225$.

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Formula

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Non-Decimal Units

Foot and Inch

Question

What is the size of a rectangle the edges of which are 4’7” and 8’9”?

Non-Decimal Units

Foot and Inch

Question

What is the size of a rectangle the edges of which are 4'7" and 8'9"?

Lösung:

4		7
8		9
sq.ft.	ft.·in.	sq.in.
32	92	63
32	7·12 + 8	63
32+7		63+8·12
39		159
40		15

Result:

5'7" · 8'9" = 48 sq.ft. 123 sq.in.

Using APL2:

```
      144 144⊤×/12⊥¨(4 7)(8 9)
40 15
```

Non-Decimal Units

Rupie and Anna

Question (1 Rupie = 16 Anas; 1 Anna = 12 Pies)

An investment of 1 Rupie yields 2 Rupies 5 Anas, what is the return of invest of 4 Rupies 9 Anas?

Non-Decimal Units

Rupie and Anna

Question (1 Rupie = 16 Anas; 1 Anna = 12 Pies)

An investment of 1 Rupie yields 2 Rupies 5 Anas, what is the return of invest of 4 Rupies 9 Anas?

Solution

2		5
4		9
Rupies	Annas	Annas

Non-Decimal Units

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Solution

2		5
4		9
Rupies	Annas	Annas
8	38	$^{45}/_{16}$

Non-Decimal Units

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Solution

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4		9
Rupies	Annas	Annas
8	38	$^{45}/_{16}$
8	$2 \cdot 16 + 6$	$^{45}/_{16}$

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8	$2{\cdot}16 + 6$	$^{45}/_{16}$
8+2		$^{45}/_{16}+ 6$
10		$8 + {}^{13}/_{16}$

Result: 4 Rp.9 An. yield 10 Rp. $8\,{}^{13}/_{16}$ An.

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8+2		$^{45}/_{16} + 6$
10		$8 + {}^{13}/_{16}$

Result: 4 Rp.9 An. yield 10 Rp. $8\,^{13}/_{16}$ An.

Using APL2:

```
    16 16 16⊤×/16⊥¨(2 5)(4 9)
10 8 13
```

Periodic Decimal Fractions

Basics

Periodic Decimal Fractions

With $\gcd(n, 10) = 1$ the period of $^1/_n$ starts immediately after the decimal point.

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Proof:

10 is a unit in $\mathbb{Z}/_{n\mathbb{Z}}$, so $n^{\varphi(n)} = 1$ with $\varphi(n) = \left|\left(\mathbb{Z}/_{n\mathbb{Z}}\right)^\times\right|$. □

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Theorem

The last digit of the decimal fraction $\frac{1}{n} = 0.\overline{a_1 a_2 \dots a_k}$ *multiplied with the last digit of n is* $9 \mod 10$*:* $a_k \cdot n \equiv 9 \mod 10$.

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Proof:

$$\underbrace{99\ldots99}_{\text{k digits}} \cdot \frac{1}{n} = \underbrace{99\ldots99}_{\text{k digits}} \cdot 0.\overline{a_1 a_2 \ldots a_k} = \left(10^k - 1\right) \cdot 0.\overline{a_1 a_2 \ldots a_k}$$

$$= a_1 a_2 \ldots a_k.\overline{a_1 a_2 \ldots a_k} - 0.\overline{a_1 a_2 \ldots a_k}$$

$$= a_1 a_2 \ldots a_k$$

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$$\underbrace{99...99}_{\text{k digits}} \cdot \frac{1}{n} = a_1 a_2 \ldots a_k$$

$$\frac{1}{n} = \frac{a_1 a_2 \ldots a_k}{\underbrace{99...99}_{\text{k digits}}} \rightarrow \underbrace{99...99}_{\text{k digits}} = n \cdot a_1 a_2 \ldots a_k$$

□

Periodic Decimal Fractions

Basics

Theorem ()

For a fraction $^1/_n$ *with* $\gcd(10, n) = 1$ *the remainders*

$$r_0 = 1;\ r_i = 10r_{i-1} - a_i n$$

build a geometric sequence modulo n *with the factor* $10 \bmod n$.

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Proof:

$$\begin{array}{llll} 1 & \to & 10 = a_1 \cdot n + r_1 \quad \text{mit} & 10 \equiv r_1 \bmod n \\ & \to & 10 r_1 = a_2 \cdot n + r_2 \quad \text{mit} & 10 r_1 \equiv r_2 \bmod n \\ & \to & \ldots & \end{array}$$

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Example

$$\frac{1}{7} \quad : \quad 1 \rightarrow 3 \rightarrow 2 \rightarrow 6 \rightarrow 4 \rightarrow 5$$

$$\rightarrow \quad \frac{1}{7} = 0.\overline{142857}$$

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Remember the theorem: $7 \cdot 7 = 9$

Periodic Decimal Fractions

Basics

Theorem ($n^2 \equiv 9 \mod 10$)

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If $n^2 \equiv 9 \bmod 10$ *we get* $a_{i-1} \equiv n \cdot r_i \bmod 10$.

Proof:

$$r_i = 10r_{i-1} - a_{i-1} \cdot n$$

$$\rightarrow \quad r_i\ n = 10r_{i-1}n - a_{i-1} \cdot n^2 \equiv -a_{i-1} \cdot (-1) \bmod 10$$

$$\rightarrow \quad r_i\ n \equiv a_{i-1} \bmod 10$$

□

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Example

$$\frac{1}{7} \quad : \quad \not{1} \not\to 3 \to 2 \to 6 \to 4 \to 5 \to 1$$

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Periodic Decimal Fractions

More Results

Observation

$\frac{1}{7}: 1 \to 3 \to 2 \to 6 \to 4 \to 5 \; (\to : \cdot 10 \quad \leftarrow : \cdot 5 \text{ denn } 10 \cdot 5 = 50)$

$1 \to 3 \to 2 \to -1 \to -3 \to -2 \pmod 7$

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Adding the remainders of the first half to the ones of the second half yields 7.

Periodic Decimal Fractions

More Results

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Theorem

If the length of the period is even, then the sum of the remainders of first half and the second half is equal to the denominator, or more precise for larger denominators: If you take negative remainders in the second half the sums get zero.

Periodic Decimal Fractions

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If the length of the period is even, then the sum of the remainders of first half and the second half is equal to the denominator, or more precise for larger denominators: If you take negative remainders in the second half the sums get zero.

Proof: With the length l of the period we have $\left(10^{l/2}\right)^2 = 10^l = 1$ which yields $10^{l/2} = -1$. Otherwise the period would be $l/2$. □

Periodic Decimal Fractions

More Results

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If the length of the period is even, then the sum of the remainders of first half and the second half is equal to the denominator, or more precise for larger denominators: If you take negative remainders in the second half the sums get zero.

Examples: $^?/_7$

$$\begin{array}{llllllll} & (1\rightarrow)3\rightarrow & 2\rightarrow & 6\rightarrow & 4\rightarrow & 5\rightarrow & 1 \\ \frac{1}{7}=0. & 1 & 4 & 2 & 8 & 5 & 7 \\ & (2\rightarrow)6\rightarrow & 4\rightarrow & 5\rightarrow & 1\rightarrow & 3\rightarrow & 2 \\ \frac{2}{7}=0. & 2 & 8 & 5 & 7 & 1 & 4 \end{array}$$

Periodic Decimal Fractions

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Periodic Decimal Fractions

More Results

Examples: $^?/_{13}$

$$\begin{array}{llllllll} & (1\to)\,10 \to & 9 \to & 12 \to & 3 \to & 4 \to & 1 \\ \frac{1}{13} = 0. & 0 & 7 & 6 & 9 & 2 & 3 \end{array}$$

Periodic Decimal Fractions

More Results

Examples: $^?/_{13}$

$$\begin{array}{lllllll} & (1\to)\,10 & \to 9 & \to 12 & \to 3 & \to 4 & \to 1 \\ \frac{1}{13} = 0. & 0 & 7 & 6 & 9 & 2 & 3 \\ & (2\to)\ 7 & \to 5 & \to 11 & \to 6 & \to 8 & \to 2 \\ \frac{2}{13} = 0. & 1 & 5 & 3 & 8 & 4 & 6 \end{array}$$

Periodic Decimal Fractions

More Results

Examples: $^?/_{13}$

$$
\begin{array}{llllllll}
 & (1\to)\,10 & \to 9 & \to 12 & \to 3 & \to 4 & \to 1 & \\
\frac{1}{13} = 0. & 0 & 7 & 6 & 9 & 2 & 3 & \\
 & (2\to)\,7 & \to 5 & \to 11 & \to 6 & \to 8 & \to 2 & \\
\frac{2}{13} = 0. & 1 & 5 & 3 & 8 & 4 & 6 & \\
\frac{5}{13} = 0. & 3 & 8 & 4 & 6 & 1 & 5 & \text{(terminates with 5)}
\end{array}
$$

Periodic Decimal Fractions

More Results

Examples: ${}^{?}/_{13}$

	$(1\rightarrow)$	$10\rightarrow$	$9\rightarrow$	$12\rightarrow$	$3\rightarrow$	$4\rightarrow$	1	
$\frac{1}{13}=0.$		0	7	6	9	2	3	
	$(2\rightarrow)$	$7\rightarrow$	$5\rightarrow$	$11\rightarrow$	$6\rightarrow$	$8\rightarrow$	2	
$\frac{2}{13}=0.$		1	5	3	8	4	6	
$\frac{5}{13}=0.$		3	8	4	6	1	5	(terminates with 5)
$\frac{11}{13}=0.$		8	4	6	1	5	3	(terminates with 3)

Periodic Decimal Fractions   Divisor greater 10, last digit 9

$$\frac{1}{n} = 0.0a_2a_3\ldots a_{k-3}a_{k-2}a_{k-1}1\ 0a_2a_3\ldots a_{k-3}a_{k-2}a_{k-1}1\ \ldots$$

With $n = 10m - 1$ we get (Multiplication with transferring digits in 10^1-position!)

$$\begin{aligned} 1 &= \left(10m \cdot \frac{1}{n}\right) - \frac{1}{n} \\ &= (m \cdot 0).(ma_2)(ma_3)\ldots(ma_{k-3})(ma_{k-2})(ma_{k-1})m * \\ &\quad -0.0a_2a_3\ldots a_{k-3}a_{k-2}a_{k-1}1 \end{aligned}$$

and division from left to right!

Periodic Decimal Fractions Divisor greater 10, last digit 9

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which yields $a_{k-1} = m \cdot a_k$, with transfer.

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and division from left to right! which yields $a_{k-1} = m \cdot a_k$, with transfer. So we get $\frac{1}{19} = 0.052631578947368421$

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$$\frac{1}{19} = 0.052631578947368421 = 0.052631578|947368421$$

Periodic Decimal Fractions Divisor greater 10, last digit 9

Examples

1. $\frac{1}{19} = 0.052631578|947368421$. Division from left to right!
2. $\frac{1}{29} = 0.03448275862068|96551724137931$
3. $\frac{1}{39} = 0.025641$

   The first and second half are not 9- but 6-complementary.
4. $\frac{1}{49} = 0.020408163265306122448|979591836734693877551$

Periodic Decimal Fractions Divisor greater 10, last digit 9

Examples

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Periodic Decimal Fractions Divisor greater 10, last digit 9

Examples

1.
2.
3. $\dfrac{1}{39} = 0.025641$
4.

Further Examples: Two Methods!

1. $\dfrac{1}{13} = \dfrac{3}{39} = 3 \cdot 0.\overline{025641} = 0.\overline{076|923}$

   a multiplication from right by 4!
2.
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Periodic Decimal Fractions Divisor greater 10, last digit 9

Examples

1.
2.
3.
4. $\frac{1}{49} = 0.020408163265306122448|979591836734693877551$

Further Examples: Two Methods!

1.
2. $\frac{1}{7} = \frac{7}{49} = (7 \cdot 0.\overline{020408163265306122448|979591836734693877551}) = 0.\overline{142857}$
3.

Periodic Decimal Fractions Divisor greater 10, last digit 9

Further Examples: Two Methods!

1. $\frac{1}{13} = \frac{3}{39} = 3 \cdot 0.\overline{025641} = 0.\overline{076|923}$
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2. $\frac{1}{7} = \frac{7}{49} = 0.\overline{142857}$
3. $\frac{1}{23} = \frac{3}{69} = 0.\overline{04347826086|95652173913}$
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Periodic Decimal Fractions — Divisor greater 10, last digit 9

Further Examples: Two Methods!

1. $\frac{1}{13} = \frac{3}{39} = 3 \cdot 0.\overline{025641} = 0.\overline{076|923}$

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2. $\frac{1}{7} = \frac{7}{49} = 0.\overline{142857}$
3. $\frac{1}{23} = \frac{3}{69} = 0.\overline{04347826086|95652173913}$
4. $\frac{1}{17} = \frac{7}{119} = 0.\overline{05882352|94117647}$

Periodic Decimal Fractions

Divisor greater 10, last digit 9

Examples

1. $\frac{1}{19} = 0.052631578|947368421\ .$
2.
3.
4.

Further Examples

1. $\frac{1}{38} = \frac{1}{2} \cdot \frac{1}{19} = 0.026315789478684210 52\ldots =$ $0.0\overline{263157894786842105}$
2. $\frac{1}{21} = \frac{1}{3} \cdot \frac{1}{7} = \frac{1}{3} \cdot 0.\overline{142|857} = 0.\overline{047619}$

Periodic Decimal Fractions

Divisor greater 10, last digit 9

Further Examples: Two Methods!

1

2 $\frac{1}{7} = \frac{7}{49} = 0.\overline{142857}$

3

4

Further Examples

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